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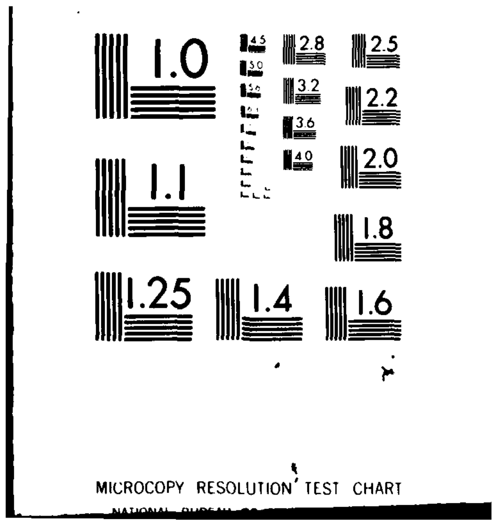
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DYNAMIC PRODUCTION NETWORKS

by

RONALD W. SHEPHARD

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DYNAMIC PRODUCTION NETWORKS

by

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APRIL 1981 ORC 81-7

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ABSTRACT

A dynamic theory of production networks is outlined with discussion of computational dynamics for such systems.

DYNAMIC PRODUCTION NETWORKS

by

Ronald W. Shephard

1. INTRODUCTION

Production systems are typically an interacting collection of production activities the outputs of which may be intermediate products serving as inputs for some activities of the system or final products, or both. The flows of intermediate products may be represented as arcs of a linear graph with activities represented by nodes. In such terms production may be modeled by a directed network connecting activities along which system exogenous inputs, intermediate and final products flow dynamically. The developments to follow are addressed to this structure of a dynamic production correspondence. See Shephard/Färe [1980] for the abstract model of a dynamic production correspondence relating histories of exogenous inputs to histories of net outputs without consideration of the network structure and intermediate products.

In the development of the abstract structure of a production network as a dynamic production correspondence, certain network axioms are needed to characterize the role of intermediate products, and to verify that the correspondence between dynamic flows of network system exogenous inputs and final outputs obeys the axioms taken for such systems and used for the network activities in treating intermediate products as activity exogenous inputs.

With this theoretical foundation one may progressively develop "ACTIVITY ANALYSIS" dynamic models from abstract to computational forms. The computational dynamics for production networks is an interesting departure from the superficial practice of indexing variables by time and

writing down the Hamilton-Jacobi differential equations for minimizing or maximizing some statement for an unstructured production model. It would seem that representing production dynamics economically by mechanical analogies is an idle practice, despite the mathematical artistry of the coordinate and variable substitutions involved in so representing the equations of motion of an arbitrary point mass system subject to forces, inertia and equations of form restricting motion.

In the discussion of computational dynamics for production networks, discrete time points will be used with step functions for histories. This approach enables use of the techniques of mathematical programming for practical purposes.

For acyclic production networks a dynamic computation of a feasible output trajectory will be outlined to illustrate the evolutionary character of "motion" for production networks, and to show possibilities for Time as well as Factor Substitution in production, an added dimension for dynamic economics.

2. ABSTRACT STRUCTURE OF DYNAMIC PRODUCTION NETWORKS

Consider a production system with N elementary activities and a final activity $(N+1)$ for recording final output rate histories. All histories are taken as time rates. As notation, the source of exogenous input rate histories is denoted by A_0. Thus the production system consists of elementary activities $A_0, A_1, A_2, \ldots, A_N, A_{N+1}$.

The primitive elements of the goods and services related to this network of production activities are time rate histories of system exogenous inputs, activity outputs, intermediate product transfers and final outputs, each defined on the nonnegative real line $\mathbb{R}_+$. Each history is an element of $(L_\infty)_+$, i.e., the nonnegative domain of equivalence classes of bounded and measurable real functions defined on $\mathbb{R}_+$. (Two functions are equivalent if they differ only on a subset of measure zero.) The norm $||f||$ of a function $f \in (L_\infty)_+$ is taken as the essential supremum. Addition, and multiplication of histories by a positive real number, are taken pointwise in time.

As notation:

$x = (x_1, x_2, \ldots, x_n) \in (L_\infty)^n_+$ is a vector of input rate histories for n system exogenous inputs (factors).

$V_i = \left(V^1_i, V^2_i, \ldots, V^m_i\right) \in (L_\infty)^m_+$, $(i = 1,2, \ldots, N)$ are vectors of net output rate histories for the N activities of the network.

$V_{ij} = \left(V^1_{ij}, V^2_{ij}, \ldots, V^m_{ij}\right) \in (L_\infty)^m_+$, $i = 1,2, \ldots, N$, are vectors of transfer rate histories from A_i to A_j as intermediate and final outputs for $j = 1,2, \ldots, N$, and $j = N+1$ respectively.

$u = (u_1, u_2, \ldots, u_m) \in (L_\infty)^m_+$ is a vector of net output rate histories for the production system.

A common space $L(_\infty)^m_+$ is taken for all activity output rate histories, transfers of the same and network output rate histories. Not all component histories of V_i , V_{ij} , u need be positive on subsets of $\mathbb{R}_+$ of positive measure.

$$||x|| = \operatorname*{Max}_i ||x_i|| \ , \ ||v_i|| = \operatorname*{Max}_k ||V^k_i|| \ ,$$

$$||V_{ij}|| = \operatorname*{Max}_k ||V^k_{ij}|| \ , \ ||u|| = \operatorname*{Max}_i ||u_i|| \ .$$

By defining the distance $\rho(f,g)$ between two histories to be the essential supremum of $|f - g|$, i.e., $\rho(f,g) = ||(|f - g|)||$, the model of production is defined by primitive elements in metric spaces. Notions of closure, convergence and limits for these elements follow naturally.

The activities of the production network each follow net output and net input dynamic production correspondences defined respectively by:

$$\left(x_{oi}, \sum_{j=1}^{N}{}' V_{ji}\right) \varepsilon\ (L_\infty)^n_+ \times (L_\infty)^m_+ \rightarrow \mathbb{P}_i\left(x_{oi}, \sum_{j=1}^{N}{}' V_{ji}\right) \varepsilon\ 2^{(L_\infty)^{2m}_+}$$

$$\left(V_{i,N+1}, \sum_{j=1}^{N}{}' V_{ij}\right) \varepsilon\ (L_\infty)^{2m}_+ \rightarrow \mathbb{L}_i\left(V_{i,N+1}, \sum_{j=1}^{N}{}' V_{ij}\right) \varepsilon\ 2^{(L_\infty)^n_+ \times (L_\infty)^m_+}$$

where x_{oi} , $i = 1,2, \ldots, N$, is an allocation to A_i from the vector x , subject to

$$\sum_{i=1}^{N} x_{oi} \leqq x \ ,$$

and $\sum{}'$ denotes a summation with the term for $j = i$ omitted.

The axioms governing the activity dynamic correspondences may be found in Shephard/Färe [1980] and will not be repeated here. Since the activities of the production network are primitive production elements for the system, they are taken to relate only exogenous input rate histories[(1)] to net output rate histories. They may range from elementary processes to factories, depending upon the extent of aggregation for the production system studied.

The production network dynamic output correspondence is expressed by:

$$x \in (L_\infty)^n_+ \to \mathbb{PN}(x) \in 2^{(L_\infty)^m_+} ,$$

$$\mathbb{PN}(x) = \left\{ u \in (L_\infty)^m_+ : u \leqq \sum_{i=1}^{N} v_{i,N+1} , \sum_{i=1}^{N} x_{oi} \leqq x , \right.$$

$$\left. \left(v_{i,N+1} , \sum_{j=1}^{N}{}' v_{ij} \right) \in \mathbb{P}_i \left(x_{oi} , \sum_{j=1}^{N}{}' v_{ji} \right) , \ i = 1,2, \ldots, N \right\} .$$

The production network dynamic input correspondence is expressed by:

$$u \in (L_\infty)^m_+ \to \mathbb{LN}(u) = \left\{ x \in (L_\infty)^n_+ : u \in \mathbb{PN}(x) \right\} \in 2^{(L_\infty)^n_+} ,$$

$$\mathbb{LN}(u) = \left\{ x \in (L_\infty)^n_+ : x \geqq \sum_{i=1}^{N} x_{oi} , \sum_{i=1}^{N} v_{i,N+1} \geqq u \right.$$

$$\left. \left(x_{oi} , \sum_{j=1}^{N}{}' v_{ji} \right) \in \mathbb{L}_i \left(v_{i,N+1} , \sum_{j=1}^{N}{}' v_{ij} \right) , \ i = 1,2, \ldots, N \right\} .$$

In these expressions, the output vector of each activity is taken net of the use by the activity of its own products. The allocation of the

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(1) Here the input rate histories exogenous to an activity may span both system exogenous inputs and intermediate product transfers.

total vector x to activities is taken freely disposable, without loss of generality. Free disposability of activity final products is not likely to be generally applicable, but is taken so here for simplicity of presentation. See Shephard [1981 forthcoming] for such complications. However, input and output histories are not taken freely disposable for the activity dynamic production correspondences. More on this in the next section.

3. NETWORK AXIOMS

It is important to investigate whether the production network correspondences $\mathbb{PN}$ and $\mathbb{LN}$ obey the properties (axioms) taken for the dynamic production correspondences of the activities of the network. The properties taken for $\mathbb{P}_i$, $\mathbb{L}_i$ do not propagate for $\mathbb{PN}$ and $\mathbb{LN}$ without some properties (axioms) postulated for production networks, since intermediate products are involved. The network axioms are:

$\mathbb{PN}$.1 For each component history $u_i \in (L_\infty)_+$ of $u \in (L_\infty)_+^m$ which is not entirely intermediate product, there exists a vector $x \in (L_\infty)_+^n$ such that $u \in \mathbb{PN}(x)$ with $||u_i|| > 0$.

$\mathbb{PN}$.2 For each activity A_i there exists a subset $\{i_1, i_2, \ldots, i_k\}$, $1 \leqq k \leqq n$, of the n exogenous input histories x_{oi} which is *globally essential* and *strong limitational* for the output vector V_i, i.e., x_{oi_j} has to be essentially positive, $j = 1,2, \ldots, k$, for $||V_i|| > 0$, and, if $||x_{oi_1}, x_{oi_2}, \ldots, x_{oi_k}|| \leqq B \in \mathbb{R}_{++}$, there exists for each V_i^o a positive scalar θ depending upon B and V_i^o such that $V_i \not\geqq \theta V_i^o$ (θ_j such that $(V_i)_j \not\geqq \theta_j (V_i^o)_j$, $j = 1,2, \ldots, m$) for all

$$V_i \in \mathbb{P}_i\left(x_{oi}, \sum_{j=1}^{N}{}' V_{ji}\right).$$

$\mathbb{PN}$.2S For each activity A_i, $\{i_1, i_2, \ldots, i_k\}$ is taken *globally totally essential* and *strong norm limitational*, i.e., x_{oi_j} has to be essentially positive, $j = 1,2, \ldots, k$, for any subset of V_i to be essentially positive, and there exists a scalar A depending on B and V_i such that $||V_i|| \not\geqq A$ when $||x_{oi_1}, \ldots, x_{oi_k}|| < B$, for all V_i.

$\mathbb{P}\mathbb{N}.3$ If $u \in (L_\infty)_+^m$ is summable in each component on $\mathbb{R}_+$ there exist x_{oi}, V_i, $i = 1,2, \ldots, N$, summable in each component with $\left(x_{oi}, \sum_{j=1}^{N}{}' V_{ji}\right) \in \mathbb{L}_i\left(V_{i,N+1}, \sum_{j=1}^{N}{}' V_{ij}\right)$,

$V_i = \sum_{j=1}^{(N+1)}{}' V_{ij}$ and $\sum_{i=1}^{N}{}' V_{i,N+1} \geqq u$.

$\mathbb{P}\mathbb{N}.4$ If $\bar{t}_u := \operatorname{Max}_i (\operatorname{Ess\,Sup} \{t \in \mathbb{R}_+ : u_i(t) > 0\}) < \infty$ and x, V_{ij} $(i = 1,2, \ldots, N$, $j = 1,2, \ldots, N, N+1)$ yield u, then

$$\begin{aligned} y_i(t) &= x_i(t), \quad t \in [0,\bar{t}_u) \\ y_i(t) &= 0, \quad t \in [\bar{t}_u,+\infty) \end{aligned} \qquad (i = 1,2, \ldots, n)$$

$$\begin{aligned} w_{ij}(t) &= V_{ij}(t), \quad t \in [0,\bar{t}_u) \\ w_{ij}(t) &= 0, \quad t \in [\bar{t}_u,+\infty) \end{aligned} \qquad \begin{pmatrix} i = 1,2, \ldots, N \\ j = 1,2, \ldots, N, N+1 \end{pmatrix}.$$

$\mathbb{P}\mathbb{N}.5$ For each activity A_i, $V_{i,N+1}$ and $\sum_{j=1}^{N}{}' V_{ij}$ are weakly disposable subvectors of V_i, i.e., if $\left(V_{i,N+1}, \sum_{j=1}^{N}{}' V_{ij}\right) \in \mathbb{P}_i\left(x_{oi}, \sum_{j=1}^{N}{}' V_{ji}\right)$ then $\left(\theta_1 \cdot V_{i,N+1}, \theta_2 \cdot \sum_{j=1}^{N}{}' V_{ij}\right) \in \mathbb{P}_i\left(x_{oi}, \sum_{j=1}^{N}{}' V_{ji}\right)$ for $\theta_1 \in [0,1]$, $\theta_2 \in [0,1]$.

$\mathbb{P}\mathbb{N}.6$ For each activity A_i, if $\left(x_{oi}, \sum_{j=1}^{N}{}' V_{ij}\right) \in \mathbb{L}_i\left(V_{i,N+1}, \sum_{j=1}^{N}{}' V_{ij}\right)$, $\left(\lambda x_{oi}, \mu \sum_{j=1}^{N}{}' V_{ji}\right) \in \mathbb{L}_i\left(V_{i,N+1}, \sum_{j=1}^{N}{}' V_{ij}\right)$, $\lambda \in [1,+\infty)$, $\mu \in [1,+\infty)$,

i.e., x_{oi} and $\sum_{j=1}^{N}{}' V_{ji}$ are weakly disposable vectors of input rate histories.

$\mathbb{P}\mathbb{N}.7$ For each activity A_i if $\left(V_{i,N+1}, \sum_{j=1}^{N}{}' V_{ij}\right) \varepsilon\, \mathbb{P}_i\left(x_{oi}, \sum_{j=1}^{N}{}' V_{ji}\right)$ and $\left|\left|V_{i,N+1}, \sum_{j=1}^{N}{}' V_{ij}\right|\right| > 0$, there exists for $\theta \varepsilon\, \mathbb{R}_{++}$ a positive scalar $\lambda_\theta^{(i)}$ such that

$$\theta \cdot \left(V_{i,N+1}, \sum_{j=1}^{N}{}' V_{ij}\right) \varepsilon\, \mathbb{P}_i\left(\lambda_\theta^{(i)} x_{oi}, \theta \cdot \sum_{j=1}^{N}{}' V_{ji}\right).$$

$\mathbb{E}\mathbb{N}.1$ For all vectors $V = (V_1, V_2, \ldots, V_N)$ such that u is obtainable from the related production network, the efficient subsets

$$\mathbb{E}_i(V_i) = \left\{\left(x_{oi}, \sum_{j=1}^{N}{}' V_{ji}\right) : \left(x_{oi}, \sum_{j=1}^{N}{}' V_{ji}\right) \varepsilon\, \mathbb{L}_i(V_i),\right.$$

$$\left(y_{oi}, \sum_{j=1}^{N}{}' w_{ji}\right) \notin \mathbb{L}_i(V_i)$$

$$\left.\text{for } \left(y_{oi}, \sum_{j=1}^{N}{}' w_{ji}\right) \geq \left(x_{oi}, \sum_{j=1}^{N}{}' V_j\right)\right\}^{(2)}$$

are uniformly bounded.

Concerning the first of these network axioms, since $(L_\infty)_+^m$ is taken to span all intermediate and final output histories, some component histories of u may be essentially null. Hence the need for this network axiom.

Axiom $\mathbb{P}\mathbb{N}.2$ states that the activities of the network cannot produce net output without some system exogenous inputs, and if the vector of essential system exogenous input histories is bounded in the norm, the scaling of each possible vector of output histories is bounded. The stronger version $\mathbb{P}\mathbb{N}.2S$

(2) The sign $\geq$ means that at least one component is "greater than" on a subset of $\mathbb{R}_+$ of positive measure.

replaces global essential by global totally essential, and strong limitational by strong norm limitational. See Shephard/Färe [1980] for definitions.

Axiom $\mathbb{P}\mathbb{N}.3$ is a statement for the network correspondence similar to $\mathbb{L}.T.1$ taken for the activity correspondences.

Axiom $\mathbb{P}\mathbb{N}.4$ is a statement for the network correspondence like axiom $\mathbb{L}.T.2$ assumed for the activity correspondences.

Axiom $\mathbb{P}\mathbb{N}.5$ permits scaling of activity distributions to final output independently of the scaling of vectors of intermediate product transfer histories. A similar assumption is made in $\mathbb{P}\mathbb{N}.6$ for vectors of exogenous input histories and vectors of intermediate product input histories to an activity.

Axiom $\mathbb{P}\mathbb{N}.7$ postulates that the scaling of a vector of output histories for an activity may be obtained by the same scaling of intermediate product input histories while the scaling of exogenous inputs may exhibit decreasing, constant or increasing returns to scale.

The axioms for dynamic production correspondences require that the subsets $\mathbb{E}_i(V_i)$ be bounded, and the network axiom $\mathbb{E}\mathbb{N}.1$ results in this same property for the production network as a whole.

Two topologies are considered for the metric spaces $(L_\infty)^n_+$, $(L_\infty)^m_+$, $(L_\infty)^{n+m}_+$, $(L_\infty)^{2m}$. The norm topology under the essential supremum norm, and a weak* topology for those spaces by price histories taken in L_1.

With the foregoing network axioms, the axioms for the activity dynamic production correspondences propagate for the network correspondences $\mathbb{P}\mathbb{N}$ and $\mathbb{L}\mathbb{N}$. See Shephard [1981, forthcoming]. The closure property $\mathbb{P}.5$ is the only property with complications for showing propagation. Two alternatives arise:

$\mathbb{P}.2S$, $\mathbb{P}.5$ (NORM TOPOLOGY) and $\mathbb{P}.5$ BIS for the activities, with $\mathbb{PN}.2$ and $\mathbb{PN}.5$,

$\mathbb{P}.2$, $\mathbb{P}.5$ (WEAK* TOPOLOGY) and $\mathbb{P}.5$ BIS for the activities, with $\mathbb{PN}.2$ and $\mathbb{PN}.5$.

Regarding the production network efficient subset

$$\mathbb{EN}(u) = \left\{x \in (L_\infty)^n_+ : x \in \mathbb{LN}(u) \ , \ y \notin \mathbb{LN}(u) \quad \text{for} \quad y \leq x\right\} ,$$

$\mathbb{EN}.1$ implies $\mathbb{EN}(u)$ is bounded. $\mathbb{EN}.1$ is required because there are an unbounded number of ways in which the outputs V_i $(i = 1,2, \ldots, N)$ may be composed to yield u .

4. AN ABSTRACT ACTIVITY ANALYSIS MODEL FOR DYNAMIC PRODUCTION NETWORKS

In some physical units or dimensionless terms, let $z = (z_1, z_2, \ldots, z_N) \in (L_\infty)_+^N$ denote a vector of intensity functions, stating for each activity the intensity of operation. These intensity functions are bounded by a vector $\bar{z} \in (L_\infty)_+^N$ of nonnegative intensity-bound-functions expressing the inherent limitations arising from physical limitations not otherwise reflected by the exogenous service inputs of facilities, equipment, and also by product design.

Technical coefficients are:

$$\mathbb{A} := \|\mathbb{A}^1 \mathbb{A}^2 \cdots \mathbb{A}^N\|^{TR} ,$$
$$\mathbb{A}^i := \|a_{i1} a_{i2} \cdots a_{in}\| , \ i = 1,2, \ldots, N , \ a_{ij} \in (L_\infty)_+$$

$$\bar{\mathbb{A}} := \|\mathbb{A}^1 \mathbb{A}^2 \cdots \mathbb{A}^N\|^{TR} ,$$
$$\bar{\mathbb{A}}^i := \|\bar{a}_{i1} \bar{a}_{i2} \cdots \bar{a}_{im}\| , \ i = 1,2, \ldots, m , \ \bar{a}_{ij} \in (L_\infty)_+$$

$$\mathbb{C} := \|\mathbb{C}^1 \mathbb{C}^2 \cdots \mathbb{C}^N\|^{TR} ,$$
$$\mathbb{C}^i := \|c_{i1} c_{i2} \cdots c_{im}\| , \ i = 1,2, \ldots, m , \ c_{ij} \in (L_\infty)_+$$

in some units such that $z_i \mathbb{A}^i$, $z_i \bar{\mathbb{A}}^i$ and $z_i \mathbb{C}^i$ are time rate histories of exogenous and intermediate product inputs, and time rate histories of outputs respectively. Then the production network dynamic output and input correspondences are:

$$\mathbb{PN}(x) = \left\{ u \in (L_\infty)_+^m : 0 \leqq z \leqq \bar{z} , \ z\mathbb{A} \leqq x , \ z(\mathbb{C} - \bar{\mathbb{A}}) \geqq 0 , \right.$$
$$\left. u \leqq z(\mathbb{C} - \bar{\mathbb{A}}) \right\}$$

$$\mathbb{LN}(u) = \left\{ x \in (L_\infty)_+^n : 0 \leqq z \leqq \bar{z} , \ z(\mathbb{C} - \mathbb{A}) \geqq u , \ x \geqq z\mathbb{A} \right\} .$$

For these formulations certain assumptions are made concerning the technical coefficients:

For each activity some exogenous input is required during $[0,+\infty)$ except on subsets of measure zero.

Each exogenous input is required by some activity during $[0,+\infty)$ on subsets of positive measure.

Each activity can produce some output on a subset of $[0,+\infty)$ of positive measure.

Each output is produced by some activity on a subset of $[0,+\infty)$ of positive measure.

The technical coefficients $\bar{a}_{ij}$ are merely taken as nonnegative as stated.

The input and output histories have been taken freely disposable for simplicity of expression. Extensions for limited disposability will be given in Shephard [1981, forthcoming].

As abstract statements these two dynamic production correspondences do not provide one with computational systems, but point the way toward such dynamic structures for production theory, expressing the inherent structure of production dynamics.

5. A COMPUTATIONAL DYNAMICS FOR LEONTIEF-LIKE PRODUCTION NETWORKS

Partition the nonnegative real line into half open intervals

$$[t-1,t) \ , \ t = 1,2,3, \ \ldots \ .$$

Take the intensity functions $z = (z_1, z_2, \ldots, z_N)$ constant on each interval. Thus among the functions $(L_\infty)_+$ only step function intensity histories are considered. This restriction is natural for a computational system. The unit of time is arbitrary. In the same way, histories of system exogenous inputs available will be taken as step functions.

At this juncture it is convenient to introduce more detail than previously considered. Exogenous inputs may be storable as well as non-storable, but not both. In order to accommodate this fact the notation for exogenous input histories is expanded to

$$\begin{array}{ll} x \ \varepsilon \ (L_\infty)^s & \text{STORAGE} \\ y \ \varepsilon \ (L_\infty)^{n-s} & \text{NONSTORAGE.} \end{array}$$

Initial inventories of storable exogenous inputs are included in the system exogenous input histories

$$x_i(0) \ , \ i = 1,2, \ \ldots, \ s \ .$$

Concerning intermediate products, initial inventories of the same are denoted by

$$v_k^o \ , \ k = 1,2, \ \ldots, \ m$$

shared by all activities of the network, and capacities for storage of intermediate products are denoted by

$$\sigma_k(t) \qquad k = 1,2, \ldots, m \ , \ t = 1,2,3, \ldots$$

independent of activity producing. Intermediate and final outputs for production during $[t-1,t)$ are taken to be realized at t. The intensity functions $z_i(t)$ and intermediate product inputs are applied at $(t-1)$ for $[(t-1),t)$.

Since some activities may yield outputs which are both intermediate product and final product, the intensity functions $z \in (L_\infty)_+^N$ are bifurcated as

$Iz_i \in (L_\infty)_+$, intermediate product production

$Fz_i \in (L_\infty)_+$, final product production.

In these terms the computational dynamics of the production network is subject to the following constraints: (See Shephard et al. [1977])

(1) $z_i(t) = Iz_i(t) + Fz_i(t)$, $z_i(t) \leqq \bar{z}_i(t)$,

$Iz_i(t) \geqq 0$, $Fz_i(t) \geqq 0$, $(i = 1,2, \ldots, N)$, $(t = 0,1,2, \ldots)$

(2) $\sum_{\tau=0}^{t} \sum_{i=1}^{N} a_{ij}(\tau) z_i(\tau) \leqq \sum_{\tau=0}^{t} x_j(\tau)$, $(j = 1,2, \ldots, s)$, $(t = 0,1,2, \ldots)$

(3) $\sum_{i=1}^{N} a_{ij}(t) z_i(t) \leqq y_j(t)$, $(j = s+1, s+2, \ldots, n)$, $(t = 0,1,2, \ldots)$

(4) $\sum_{i=1}^{N} \sum_{\tau=0}^{t} \bar{a}_{ik}(\tau) z_i(\tau) \leqq v_k^o + \sum_{i=1}^{N} \sum_{\tau=0}^{t-1} c_{ik}(\tau+1) Iz_i(\tau)$,

$\sum_{i=1}^{N} \bar{a}_{ik}(0) z_i(0) \leqq v_k^o$, $(k = 1,2, \ldots, m)$, $(t = 1,2,3, \ldots)$

(5) $v_k^o + \sum_{i=1}^{N} \sum_{\tau=0}^{t-1} c_{ik}(\tau+1) Iz_i(\tau) - \sum_{i=1}^{N} \sum_{\tau=0}^{t} \bar{a}_{ik}(\tau) z_i(\tau) \leqq \sigma_k(t)$,

$v_k^o - \sum_{i=1}^{N} \bar{a}_{ik}(0) z_i(0) \leqq \sigma_k(0)$, $(k = 1,2, \ldots, m)$, $(t = 1,2,3, \ldots)$.

The constraints (1) limit the intensities of operation of the activities of the production network to be nonnegative and not to exceed certain physical limitations not otherwise expressed by the service components of the exogenous input histories.

The constraints (2) and (3) limit exogenous inputs to available resources as given by the time histories $x \in (L_\infty)^s_+$ and $y \in (L_\infty)^{n-s}_+$. The constraints (4) require that inputs of intermediate products by the network activities do not exceed the supply available from the outputs of the activities and initial inventories. The last set of constraints does not allow accumulation of inventories of intermediate products beyond capacities for the same. All these constraints are limitations on network activity intensities.

The foregoing constraint system does not in any way predetermine the allocation of resources. However, it is a general basis for a computational dynamics of Leontief-like production networks. But since nothing has been specified concerning final outputs the dynamic evolution of the production network is not directed. The intensity functions Fz_i , $(i = 1,2, \ldots, N)$ are so to speak free elements.

Since the output set $\mathbb{P}N(x,y)$ in general may be taken to exhibit weak disposability for vectors $u \in (L_\infty)^m_+$ of output histories, one may generate for various feasible output mixes of output histories the maximal scalar extension in $\mathbb{P}N(x,y)$, i.e., determine the dynamic evolution of the system to find points on the frontier of $\mathbb{P}N(x,y)$. Also for some price histories of the various outputs and inputs, one may seek to control the dynamics of the system to maximize (Revenue-Cost of Resources). In either case it is convenient to make these generations of the dynamics of the system with respect to a finite planning horizon T .

Now suppose there are $1 \leqq P \leqq m$ net products possible for the system. Let

$$\bar{a}_{N+1,p} \in \mathbb{R}_+$$

be coefficients related to an index

$$z_{N+1} \in \mathbb{R}_+$$

such that

$$\bar{a}_{N+1,p} \quad z_{N+1} \ , \ p = 1,2, \ldots, P$$

defines the total amount of the p^{th} final product accumulated over the planning interval $[0,T]$. The coefficients $\bar{a}_{N+1,p}$ may be chosen to determine a specified output mix. Then, in order to direct the system this way one adds the constraint

$$(6) \quad \bar{a}_{N+1,p} \cdot z_{N+1} \leqq \sum_{i=1}^{N} \sum_{t=1}^{T} c_{ip}(t) F z_i(t-1) \ , \ (p = 1,2, \ldots, P)$$

and determines the dynamic evolution of the system by the following linear program:

$$\begin{cases} \text{Max} & z_{N+1} \\ \text{Subject to} & z_{N+1} \geqq 0 \ , \ (1),(2),(3), \ldots, (5),(6) \quad \text{on} \quad [0,T] \ . \end{cases}$$

A calculation of this type has been outlined by Leachman [1980] for the case of a network with each activity producing a single output, possibly with cycles in the network, i.e., for a Leontief-like network. Also similar calculations were outlined for this particular network to minimize the cost of obtaining a given program of output histories.

It is clear from the foregoing that productivity of a production system is a complicated concept. For any given output mix one may determine the maximal throughput for the mix, but this will vary with output mix. The resource inputs likewise condition the throughput. How is one then to determine properly the maximal productivity potential of a system unless resources and output mix are balanced for this purpose? Obviously a good deal of study is needed for such questions. With declining energy resources it would be of interest to know how the maximal throughput of various output mixes would retrogress.

In the general terms of the formulation of production networks used here, real capital is expressed in terms of service input rates, and for those components of (x,y) which are fully utilized in the dynamic solution of the linear program and reflect real capital, one may make marginal analyzes (linear programs) for incremental changes in such inputs. Indeed for energy supply decreases one may seek to estimate the consequences for economic sectors by the dynamic Leontief model detailed by Leachman, using "constant dollar values" for measures of aggregate output.

In order to drive the dynamics of the system by maximizing total revenue minus cost of resources (x,y) applied, one need merely define for given price histories $r_p(t)$, $q_j(t)$ the objective function

$$\Pi(T) = \sum_{i=1}^{N} \left(\sum_{t=0}^{T-1} r_p(t+1) c_{ip}(t+1) Fz(t) \right) - \sum_{\tau=0}^{T-1} \left(\sum_{j=1}^{s} q_j(\tau) x_j(\tau) - \sum_{j=s+1}^{n} q_j(\tau) y_j(\tau) \right)$$

and determine the dynamic evolution of the system by the linear program:

$$\begin{cases} \text{Max} & \Pi(T) \\ \text{Subject to} & (1),(2),(3),\ \ldots,\ (5) \ \text{ on } \ [0,T] \ . \end{cases}$$

Here $x_j(\tau)$, $y_j(\tau)$ are also variables in the linear program. Obviously refinements can be made in this kind of generation of the dynamics of the production network by discounting and adding other costs to complicate the definition of profit. For the purposes of this paper, interest is mainly in structure, and not management practice.

6. ACYCLIC PRODUCTION NETWORKS

In a production network without cycles the activities of the network may be ordered so that the outputs of an activity A_i can serve as intermediate product only for $A_{i+1}, A_{i+2}, \ldots, A_N$, or what is the same, intermediate product inputs to A_i can arise only from $A_1, A_2, \ldots, A_{i-1}$. Then, for preassignment of (1) resources, (2) preallocation of distributing intermediate products and (3) preallocation of activity outputs to final product, dynamic trajectories may be calculated (evolved) for the intensity functions of the activities and thereby determine the time histories of intermediate and final products. See Shephard/et al. [1977] for such calculations for shipbuilding.

For the general model of computational dynamics for production networks, let

$$x_{oi} = (x_{oi1}, x_{oi2}, \ldots, x_{ois}) \qquad i = 1,2, \ldots, N$$

$$y_{oi} = (y_{oi1}, y_{oi2}, \ldots, y_{oi(n-s)}) \qquad i = 1,2, \ldots, N$$

denote a preallocation of exogenous input histories to the activities, with

$$\sum_{i=1}^{N} x_{oi} \leqq x \ , \ \sum_{i=1}^{N} y_{oi} \leqq y \ .$$

Further, the ordering taken for the network is such that $\bar{A}_i$ takes the form

$$||\bar{a}_{i1}\bar{a}_{i2} \cdots \bar{a}_{i,i-1}\ , 0 \cdots 0|| \qquad i = 1,2, \ldots, N \ .$$

Let

$$\delta^{N+1}_{j,k}(t)\ ,\ (j = 1,2, \ldots, N)\ ,\ (k = 1,2,3, \ldots, m)\ ,\ (t = 1,2,3, \ldots)$$

denote the fraction of the k^{th} output rate history of A_j going to net output at the time t. These preassignments satisfy

$$0 \leqq \delta_{j,k}^{N+1}(t) \leqq 1 \ , \ (t = 1,2,3, \ldots) \ , \ (k = 1,2, \ldots, m) \ , \ (j = 1,2, \ldots, N)$$

$$\sum_{j=1}^{N} \delta_{j,k}^{N+1}(t) > 0 \ , \text{ for some } k \in \{1,2, \ldots, m\} \ , \ t \in \{1,2,3, \ldots\}$$

and the net output histories $u \in (L_\infty)_+^m$ are given by

$$u_k(t) = \sum_{j=1}^{N} \delta_{j,k}^{N+1}(t) Z_j(t-1) c_{j,k}(t) \ , \ (t = 1,2,3, \ldots) \ , \ (k = 1,2, \ldots, m) \ .$$

Let

$$1 \geqq \Delta_{ijk}(t) \geqq 0 \ , \ (k = 1,2, \ldots, m) \ , \ (j = 1,2, \ldots, N-1) \ , \ (i = (j+1),(j+2), \ldots, N)$$

denote a preallocation of the k^{th} output rate history of A_j at time t to the activities A_i, $(i = (j+1),(j+2), \ldots, N)$ as intermediate product. These coefficients satisfy

$$\sum_{i=j+1}^{N} \Delta_{ijk}(t) = 1 - \delta_{j,k}^{N+1}(t) \ , \ (k = 1,2, \ldots, m) \ , \ (t = 1,2,3, \ldots)$$
$$(j = 1,2, \ldots, (N-1)) \ .$$

Since intermediate product time histories are preallocated, initial inventories and capacities for storing intermediate products are now distinguished by activity as:

$$v_{ik}^{o} \quad (i = 1,2, \ldots, N) \ , \ (k = 1,2, \ldots, m)$$

$$\sigma_{ik}(t) \quad (i = 1,2, \ldots, N) \ , \ (k = 1,2, \ldots, m) \ , \ (t = 0,1,2,3, \ldots) \ .$$

Note that

$$v_{ik}^{o} \leqq \sigma_{ik}(0) \ .$$

Certain simplifying assumptions may be taken for the coefficients $\bar{a}_{ik}(\tau)$, $c_{jk}(\tau)$:

either $\bar{a}_{ik}(\tau) > 0$ for all $\tau \in [0,\infty)$ or $\bar{a}_{ik}(\tau) = 0$ for all $\tau \in [0,\infty)$, $(i = 1,2, \ldots, N)$

either $c_{jk}(\tau) > 0$ for all $\tau \in [0,+\infty)$ or $c_{jk}(\tau) = 0$ for all $\tau \in [0,+\infty)$, $(j = 1,2, \ldots, N)$.

In other words, an activity is consistent in the use of intermediate products and production of outputs. This does not preclude alternative activities with different intermediate product inputs and different output commodities. If an activity cannot satisfy the two assumptions it may be subdivided until it does.

Also, with little if any loss of generality one may assume

$$\sigma_{ik}(t) \geqq \sigma_{ik}(t-1) \ , \ (t = 1,2, \ldots) \ , \ (i = 1,2, \ldots, N) \ , \ (k = 1,2, \ldots, m) \ .$$

By definition

$$\Delta_{ijk}(\tau) = 0 \ \text{ if } \ \bar{a}_{ik}(\tau) = 0 \ .$$

One does not transfer intermediate product output to an activity not using it.

Then the previous constraints for the dynamic system take the following form:

$$(1)^* \quad 0 \leqq z_i(t)\bar{z}_i(t) \ , \ (t = 0,1,2,3, \ldots) \ , \ (i = 1,2, \ldots, N)$$

$$(2)^* \quad \sum_{\tau=0}^{t} a_{ik}(\tau)z_i(\tau) \leqq \sum_{\tau=0}^{t} x_{oik}(\tau) \ , \ (k = 1,2, \ldots, s) \ ,$$
$$(i = 1,2, \ldots, N) \ , \ (t = 0,1,2, \ldots)$$

$$(3)^* \quad a_{ik}(t)z_i(t) \leqq y_{oik}(t) \ , \ (k = (s+1),(s+2), \ldots, n) \ ,$$
$$(i = 1,2, \ldots, N) \ , \ (t = 0,1,2, \ldots)$$

$$(4)^* \quad \sum_{\tau=0}^{t} \bar{a}_{ik}(\tau)z_i(\tau) \leqq v_{ik}^{o} + \sum_{\tau=1}^{t} \sum_{j=1}^{(i-1)} \Delta_{ijk}(\tau)c_{j,k}(\tau)z_j(\tau-1)$$
$$\bar{a}_{ik}(0)z_i(0) \leqq v_{ik}^{o} \ , \ (t = 1,2,3, \ldots) \ , \ (i = 2,3,4, \ldots, N) \ ,$$
$$(k = 1,2,3, \ldots, m)$$

$$(5)^* \quad v_{ik}^{o} + \sum_{\tau=1}^{t} \sum_{j=1}^{(i-1)} \Delta_{ijk}(\tau)c_{jk}(\tau)z_j(\tau-1) - \sum_{\tau=0}^{t} \bar{a}_{ik}(\tau)z_i(\tau) \leqq \sigma_{ik}(\tau)$$
$$(i = 1,2, \ldots, N) \ , \ (k = 1,2, \ldots, m) \ , \ (t = 1,2,3, \ldots) \ .$$

Notice now that the intensity functions need not be separated into Iz , Fz due to the coefficients $\delta_{j,k}^{N+1}(t)$, $(t = 1,2, \ldots)$.

It is of some interest to consider the dynamic development of the trajectories for final outputs from given preallocations. The inequality system $(1)^*,(2)^*, \ldots, (5)^*$ may be used to develop trajectories for the intensity functions $z_i(t)$, $t = 0,1,2, \ldots$ which generates the histories of all outputs, both intermediate and final. A policy of applying feasible intensities $z_i(t)$ is needed in order to get a specific trajectory. It is convenient to seek to take this policy as: "utilize the maximal value of $z_i(t)$ possible for each $t = 0,1,2, \ldots$," i.e., a Greedy Policy. This policy ignores the problem of "variable loading." However, it will enable one to observe that dynamic systems of production involve Time Substitution as an additional dimension beyond those for static or steady state systems.

The Greedy policy for this system is generated as follows.

Define:

$$S^{(i)}(t) = \{k \in \{1,2, \ldots, s\} : a_{ik}(t) > 0\}$$

$$\tilde{S}^{(i)}(t) = \{k \in \{s+1, s+2, \ldots, n\} : a_{ik}(t) > 0\}$$

$$\textstyle\sum^{(i)}(t) = \{k \in \{1,2, \ldots, m\} : \bar{a}_{ik}(t) > 0\} .$$

Define:

$$R^{(i)}(0) = \underset{k \in S^{(i)}(0)}{\mathrm{Min}} \left\{ \frac{x_{oik}(0)}{a_{ik}(0)} \right\} \quad \text{if} \quad S^{(i)}(0) \neq \emptyset$$

$$= +\infty \qquad \text{if} \quad S^{(i)}(0) = \emptyset$$

$$R^{(i)}(t) = \underset{k \in S^{(i)}(t)}{\mathrm{Min}} \left\{ \frac{\sum_{\tau=0}^{t} x_{oik}(\tau) - \sum_{\tau=0}^{t-1} a_{ik}(\tau) z_i(\tau)}{a_{ik}(t)} \right\} \quad \text{if} \quad S^{(i)}(t) \neq \emptyset$$

$$= +\infty \qquad \text{if} \quad S^{(i)}(t) = \emptyset$$

$$\tilde{R}^{(i)}(t) = \underset{k \in \tilde{S}^{(i)}(t)}{\mathrm{Min}} \left\{ \frac{y_{oik}(t)}{a_{ik}(t)} \right\} \quad \text{if} \quad \tilde{S}^{(i)}(t) \neq \emptyset$$

$$= +\infty \qquad \text{if} \quad \tilde{S}^{(i)}(t) = \emptyset$$

$$W^{(i)}(t) = \underset{k \in \sum^{(i)}(t)}{\mathrm{Min}} \left\{ \frac{v^{o}_{ik} + \sum_{\tau=1}^{t} \sum_{j=1}^{(i-1)} \Delta_{ijk}(\tau) c_{jk}(\tau) z_j(\tau - 1) - \sum_{\tau=1}^{t-1} \bar{a}_{ik}(\tau) z_i(\tau)}{\bar{a}_{ik}(t)} \right\} \quad \text{if} \quad \textstyle\sum^{(i)}(t) \neq \emptyset$$

$$= +\infty \qquad \text{if} \quad \textstyle\sum^{(i)}(t) = \emptyset .$$

Define

$$Q_i(t) = \text{Min}\ \{R^{(i)}(t),\tilde{R}^{(i)}(t),W^{(i)}(t),\bar{Z}_i(t)\}\ .$$

Then the Greedy Policy is

$$Z_i(t) = Q_i(t) \qquad (i = 1,2,\ \ldots,\ N)\ ,\ (t = 1,2,3,\ \ldots)$$

if constraint $(5)^*$ is satisfied. However, $(5)^*$ may not be satisfied at some time $t_o > 0$ for one or more commodities k and activity i. Let k_o yield the maximal value

$$v^o_{ik_o} + \sum_{\tau=1}^{t_o} \sum_{j=1}^{(i-1)} \Delta_{ijk_o}(\tau)c_{jk_o}(\tau)z_j(\tau-1) - \sum_{\tau=0}^{t_o} \bar{a}_{ik_o}(\tau)z_i(\tau) - \sigma_{ik_o}(t_o) > 0\ .$$

Presumably $(5)^*$ is satisfied at (t_o-1) for i and k_o, which implies

$$\sum_{j=1}^{(i-1)} \Delta_{ijk_o}(t_o)c_{jk}(t_o)z_j(t_o-1) - \bar{a}_{ik_o}(t_o)z_i(t_o) > 0\ .$$

Evidently, $(5)^*$ may be satisfied at t_o if $z_j(\tau) = 0$ for $\tau = (t_o-1)$, $(j = 1,2,\ \ldots,\ (i-1))$. As an approximation to a Greedy Policy one may take these values for $z_j(t_o-1)$ for $(j = 1,2,\ \ldots,\ (i-1))$.
The solutions for $z_j(\tau)$, $(j = 1,2,\ \ldots,\ (i-1)$, $(\tau = t_o$, $(t_o+1),(t_o+2),\ \ldots)$ need to be updated. Then one may proceed t with the calculation for $z_i(t_o)$, $z_i(t_o+1)$, etc.

The inventory capacity bounds $\sigma_{ik}(t)$ may require considerable recalculation. However, the routine suggested is a simple one and does provide a ready policy.

The evolutionary character of the trajectory for a greedy type solution is evident. In the order $(i = 1,2,3, \ldots, N)$ one calculates $z_i(t)$ for $(t = 0,1,2,3, \ldots)$ corresponding to the given preassignments of activity outputs to other activities and final output. The net output histories for the production network are given by

$$u_k(t) = \sum_{j=1}^{N} \delta_{jk}^{(N+1)}(t) c_{jk}(t) z_j(t-1) \quad \begin{pmatrix} k = 1,2, \ldots, m \\ t = 1,2,3, \ldots \end{pmatrix} .$$

Now, insofar as the modified Greedy Policy implies for some activity intensities that periods of zero intensity occur, one may seek to smooth load such production intensities by operating them at less than maximal intensity without altering the output histories obtained over some planning period, i.e., there may be possibilities for Time Substitution.

Time substitution for the activity intensity trajectories, and through them the application of exogenous input and intermediate product histories is an added dimension for dynamic production networks over that of static or steady state models of production. The related problems of smooth loading are the gist of dynamic production planning. See Leachman [1980] for some mathematical programming methods to smooth load for shipbuilding.

Maximizing throughput as discussed in the previous section and the one to follow, and other optimizations, are not always meaningful, because preassignments of exogenous inputs and intermediate and final outputs may be required for the facilities (activities) of a large production network. Also a compounding of the complexity of an optimization may be involved to assure a reasonable smooth loading. There is some advantage to simulate a dynamic trajectory, in the face of uncertainty of information, for policy formation which can be updated from time to time.

7. A COMPUTATIONAL DYNAMICS FOR GENERALIZED PRODUCTION NETWORKS

In Sections 4, 5 and 6 above, the activities of the network were driven by intensity functions $z_i \in (L_\infty)_+$ operating on fixed technical coefficients for system exogenous inputs, intermediate product inputs and activity outputs, independently of the intended distribution of activity outputs.

Now instead of a single intensity function $z_i \in (L_\infty)_+$ for A_i, let

$$z_i := (z_{i1}, z_{i2}, \ldots, z_{iN}, z_{i,N+1}) \in (L_\infty)_+^{N+1}$$

denote an intensity vector with $z_{ij} \in (L_\infty)_+$ denoting the intensity of operating A_i for output to go to A_j, $(j = 1,2, \ldots, N, N+1)$. The system exogenous input histories implied by z_i are $z_i \mathbb{A}_i$ where

$$\mathbb{A}_i := \text{TRANSPOSE } \|\mathbb{A}_{i1} \mathbb{A}_{i2} \cdots \mathbb{A}_{iN}, \mathbb{A}_{i(N+1)}\| \;, \quad (i = 1,2, \ldots, N)$$

$$\mathbb{A}_{ij} := \|a_{ij1} a_{ij2} \cdots a_{ijn}\| \;, \quad (j = 1,2, \ldots, N, N+1) \;.$$

Similarly, intermediate product inputs are given by $z_i \bar{\mathbb{A}}_i$ where

$$\mathbb{A}_i := \text{TRANSPOSE } \|\bar{\mathbb{A}}_{i1} \bar{\mathbb{A}}_{i2} \cdots \bar{\mathbb{A}}_{iN} \bar{\mathbb{A}}_{i(N+1)}\| \;, \quad (i = 1,2, \ldots, N)$$

$$\bar{\mathbb{A}}_{ij} := \|\bar{a}_{ij1} \bar{a}_{ij2} \cdots \bar{a}_{ijn}\| \;, \quad (j = 1,2, \ldots, N, N+1)$$

and activity outputs are given by $z_i \mathbb{C}_i$ $(i = 1,2, \ldots, N)$ where

$$\mathbb{C}_i := \text{TRANSPOSE } \|\mathbb{C}_{i1} \mathbb{C}_{i2} \cdots \mathbb{C}_{iN} \mathbb{C}_{i(N+1)}\| \;, \quad (i = 1,2, \ldots, N)$$

$$\mathbb{C}_{ij} := \|c_{ij1}, c_{ij2} \cdots c_{ijn}\| \;, \quad (j = 1,2, \ldots, N, N+1) \;.$$

The constraints for the production network are:

$$(1)^{**} \quad 0 \leqq \sum_{j=1}^{N+1} z_{ij}(t) \leqq \bar{z}_i(t) \in (L_\infty)_+ , \begin{pmatrix} i = 1,2, \ldots, N \\ t = 0,1,2, \ldots \end{pmatrix}$$

$$(2)^{**} \quad \sum_{i=1}^{N} \sum_{j=1}^{N+1} \sum_{\tau=0}^{t} a_{ijk}(\tau) z_{ij}(\tau) \leqq \sum_{\tau=0}^{t} x_k(\tau) , \begin{pmatrix} k = 1,2, \ldots, s \\ t = 0,1,2, \ldots \end{pmatrix}$$

$$(3)^{**} \quad \sum_{i=1}^{N} \sum_{j=1}^{N+1} a_{ijk}(t) z_{ij}(t) \leqq y_k(t) , \begin{pmatrix} k = (s+1), \ldots, n) \\ t = 0,1,2, \ldots \end{pmatrix}$$

$$(4)^{**} \quad \sum_{j=1}^{N+1} \sum_{\tau=0}^{t} \bar{a}_{ijk}(\tau) z_{ij}(\tau) \leqq v_{ik}^{o} + \sum_{j=1}^{N} \sum_{\tau=1}^{t} c_{jik}(\tau) z_{ji}(\tau - 1)$$

$$\sum_{j=1}^{N+1} \bar{a}_{ijk}(0) z_{ij}(0) \leqq v_{ik}^{o} , \quad (i = 1,2,3, \ldots, N) ,$$

$$(k = 1,2, \ldots, m) , \quad (t = 1,2,3, \ldots)$$

$$(5)^{**} \quad v_{ik}^{o} + \sum_{j=1}^{N} \sum_{\tau=1}^{t} c_{jik}(\tau) z_{ji}(\tau - 1) - \sum_{j=1}^{N+1} \sum_{\tau=0}^{t} \bar{a}_{ijk}(\tau) z_{ij}(\tau) \leqq \sigma_{ik}(t)$$

$$(i = 1,2,3, \ldots, N) , \quad (k = 1,2, \ldots, m) , \quad (t = 1,2,3, \ldots) .$$

The "motion" of this system for a production plan needs direction by optimizing some objective. Maximum throughput of a given output mix over a planning period T can be used to direct this production system by the following linear program

Max Z_{N+1}

Subject to: $Z_{N+1} \geqq 0$, $(1)^{**}$, $(2)^{**}$, $(3)^{**}$, $(4)^{**}$, $(5)^{**}$, $(6)^{**}$ on $[0,T]$

where

$$(6)^{**} \quad \bar{a}_{N+1,k} z_{N+1} \leqq \sum_{i=1}^{N} \sum_{t=1}^{T} c_{i,N+1,k}(\tau) z_{i,N+1}(\tau - 1) , \quad (k = 1,2, \ldots, P) .$$

The conventions of Section 5 are used in this formulation.

The minimal time horizon for a production plan to yield given accumulations

$$w_k \;,\; (k = 1,2,\; \ldots,\; P)$$

of the final products can be found by replacing (6)** by

$$(7)\quad w_k \leq \sum_{i=1}^{N} \sum_{\tau=1}^{T} c_{i,N+1,k}(\tau) z_{i,N+1}(\tau - 1) \qquad (k = 1,2,\; \ldots,\; P)$$

and consecutively seeking a feasible solution to (1)**, (2)**, (3)**, (4)**, (5)**, (7) for increasing $T = 1,2,3, \ldots$ until feasibility is first attained. See Leachman [1980] for this computation for production networks with activities each of which produce a single output.

Other optimizations are possible to drive the "motion" of the production system; and additional constraints may be significant. For example, large changes of service inputs may involve learning and shift the system retrogressively to higher amounts required per unit intensity.

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